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Interface Sharpness in the Ising Model with Long-Range Interaction

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For the 3-dimensional Ising model with long-range interaction, Gibbs states are constructed that are small perturbations of non-translation-invariant ground states. These ground states are in one-to-one correspondence with the set of all rational planes.

KEY WORDS: Ground state; Gibbs state; interface sharpness; Kirkwood–Salzburg equation; contour model with interaction.

1. FORMULATION OF RESULTS

We consider the Ising ferromagnet on the lattice with long-range interaction. The Hamiltonian has the form

$$H(\varphi) = -\frac{1}{2} \sum_{x, y \in Z^3} J(x-y)\, \varphi(x)\, \varphi(y) \tag{1}$$

where the spin variables $\varphi(x)$ take the values ± 1, the potential $J(x-y)$ is a nonnegative, translation-invariant function, and $\sum_{x \in Z^3} J(x) < \infty$.

The periodic ground states of the Ising ferromagnet are $\varphi^+(x) \equiv 1$, $\varphi^-(x) \equiv -1$, $x \in Z^3$.[1] There are also non-translation-invariant ground-state configurations. The set of some of them is described by the following theorem:

Theorem 1. Let π denote the plane given by the equation

$$r_1 x_1 + r_2 x_2 + r_3 x_3 = 0 \tag{2}$$

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If $r_\pi = (r_1, r_2, r_3)$, then the configuration

$$\varphi_\pi(x) = \begin{cases} -1, & (r_\pi \cdot x) \geq 0 \\ 1, & (r_\pi \cdot x) < 0 \end{cases} \tag{3}$$

is a ground state of the model (1).

Geometrically, Theorem 1 means the following. Given any plane π, the ground state $\varphi_\pi(x)$ is the configuration having different values of spins in the different halves of $\mathbf{R}^3$ defined by π.

Remark. Theorem 1 remains true in any dimension.

The model (1) was considered by Burkov,[2] who proved Theorem 1 for the case of a two-dimensional lattice by a method based on the Hubbard criterion.

In this paper we study the structure of the set of low-temperature Gibbs states of the model (1). Griffiths inequalities imply that when the temperature is low, the model (1) has at least two extremal Gibbs states (from the fact that the usual Ising ferromagnet has). Let us denote these states by P^+ and P^-. They are small perturbations of the ground states $\varphi^+(x)$ and $\varphi^-(x)$. We consider the question of the existence of Gibbs states that are small perturbation of the configurations $\varphi_\pi(x)$ for various planes π. Dobrushin[3] was the first to discover the existence of nonperiodic Gibbs states connected with nonperiodic ground states in the Ising ferromagnet.

We call a plane π rational if r_i and r_j are rationally dependent numbers for all pairs (i, j), $i, j = 1, 2, 3$. It is called semirational if only one of three pairs of numbers r_i, r_j are rationally dependent, $i \neq j$. Other planes are called irrational. In the case of a rational π, the π interface of the model (1) and the (1, 0, 0) interface of the Ising ferromagnet have similar properties. Thus, it is reasonable to expect the existence of a stable interface separating ferromagnetic phases in model (1). Under some additional assumption this is really true: the main result of the present paper is the following.

Theorem 2. Assume that $J(x-y) = J(r) = r^{-\alpha}$, $\alpha > 9$, and assume π to be an arbitrary rational plane. Then one can find $\beta_\pi^{\mathrm{cr}} = \mathrm{const}/J_\pi$ such that if $\beta > \beta_\pi^{\mathrm{cr}}$, then there exists a Gibbs state P_π such that

$$P_\pi(\varphi(x) = -1; x\colon \varphi_\pi(x) = -1) > 1 - g(\beta) \tag{4}$$

$$P_\pi(\varphi(x) = -1; x\colon \varphi_\pi(x) = 1) \leq g(\beta) \tag{5}$$

where $g(\beta) \to 0$ when $\beta \to \infty$ and J_π is constant depending on the plane π only [defined just before Eq. (10) below].

An infinite set of Gibbs states P_π^i is obtained by parallel shifts of the plane π.

Theorem 2 easily leads to the following result:

Theorem 3. Assume that $J(r) = r^{-\alpha}$, $\alpha > 9$, and some temperature $\hat{\beta}^{-1}$ is fixed. Then the Gibbs states describing the coexistence of phases exist for any rational plane such that $\beta_\pi^{\text{cr}} < \hat{\beta}$.

Theorems 2 and 3 can be easily generalized to any lattice Z^ν, $\nu > 3$. The condition $\nu \geqslant 3$ is essential. Recently it has been shown[(4–6)] that all low-temperature Gibbs states are translation invariant for a wide class of models on Z^2.

Now we explain briefly the scheme of the proof of Theorem 2. The main strategy is similar to that in ref. 3. The main differences are as follows. The new point is that we have to investigate the model with long-range interaction. So it is necessary to consider thick contours and contour models with interaction. This leads to some modifications in the definitions of walls and ceilings. In addition we develop a special technique for decomposing the partition function (Section 4).

2. INVESTIGATION OF GROUND STATES

In this section the proof of Theorem 1 will be given.

Proof of Theorem 1. Let the plane π be fixed and the configuration $\varphi_\pi(x)$ be defined by (3). To prove the statement, we have to show that

$$H(\tilde{\varphi}_\pi(x) \mid \varphi_\pi(x)) = H(\tilde{\varphi}_\pi(x)) - H(\varphi_\pi(x)) \geqslant 0 \tag{6}$$

where the configuration $\tilde{\varphi}_\pi(x)$ is a local perturbation of the configuration $\varphi_\pi(x)$ on an arbitrary, finite set $A \subset Z^3$. Note that $H(\tilde{\varphi}_\pi(x) \mid \varphi_\pi(x))$ is finite, since $\sum_{x \in Z^3} J(x) < \infty$ and the perturbation is finite. By definition

$$H(\tilde{\varphi}_\pi(x)) - H(\varphi_\pi(x)) = \frac{1}{2} \sum_{x,\, y \in Z^3} J(x-y)(\varphi_\pi(x)\,\varphi_\pi(y) - \tilde{\varphi}_\pi(x)\,\tilde{\varphi}_\pi(y)) \tag{7}$$

Let us decompose the set of all pairs (x, y), $x, y \in Z^3$, into the following four classes:

$$M_1 = \{(x, y){:}\ x \in A,\ y \in A\}$$

$$M_2 = \{(x, y){:}\ x \notin A,\ y \notin A\}$$

$$M_3 = \{(x, y){:}\ \varphi_\pi(x)\,\varphi_\pi(y) = 1,\ x \in A,\ y \notin A, \text{ or } x \notin A,\ y \in A\}$$

$$M_4 = \{(x, y){:}\ \varphi_\pi(x)\,\varphi_\pi(y) = -1,\ x \in A,\ y \notin A, \text{ or } x \notin A,\ y \in A\}$$

Then the formula (7) can be written as follows:

$$H(\tilde{\varphi}_\pi(x)) - H(\varphi_\pi(x)) = \sum_{(x,y)\in M_1} \alpha(x,y) + \sum_{(x,y)\in M_2} \alpha(x,y) + \sum_{(x,y)\in M_3} \alpha(x,y) + \sum_{(x,y)\in M_4} \alpha(x,y) \quad (8)$$

where

$$\alpha(x,y) = \tfrac{1}{2} J(x-y)[\varphi_\pi(x)\,\varphi_\pi(y) - \tilde{\varphi}_\pi(x)\,\tilde{\varphi}_\pi(y)] \quad (9)$$

Now note that all the terms $\alpha(x, y)$ vanish in the first and second sums, all the terms are greater than or equal to zero in the third sum, and all the terms $\alpha(x, y)$ are smaller than or equal to zero in the fourth sum. Thus, to prove the inequality (6) it is sufficient to verify that for each negative term of the fourth sum it is possible to find a unique positive term of the third sum such that their absolute values are equal. Suppose $(\tilde{x}, \tilde{y}) \in M_4$. By definition $\varphi_\pi(\tilde{x})\,\varphi_\pi(\tilde{y}) = -1$ and one of the following symmetric conditions holds: (1) $\tilde{x} \in A$, $\tilde{y} \notin A$; (2) $\tilde{x} \notin A$, $\tilde{y} \in A$.

Assume that condition 1 holds. Let us define the sequence of points v_n, $n \geq 1$, as follows: $v_1 = \tilde{y}$, $v_2 = \tilde{x}$, $v_n = T(v_{n-1}, v_{n-2})$ for $n > 2$, where $T(x, y) = 2x - y$ denotes the point that is symmetric with the point y with respect to the center x. Then we introduce the sequence of pairs $w_n = (v_n, v_{n+1})$, $n \geq 1$. By definition $w_1 \in M_4$, and $w_i \notin M_4$ for $i > 1$. Because A is finite, there exists N such that $w_i \in M_2$, $i > N$. Notice that since the sequence $\varphi(v_n)$, $n \geq 1$, contains terms with different signs, the set $\{w_k : w_k \in M_3\}$ is not empty and the number

$$m = \min_{w_k \in M_3} k$$

is well-defined. Obviously, $J(v_{i+1}, \subset v_i) = J(v_{j+1}, v_j)$ for any $i, j \geq 1$. Finally, the needed correspondence is constructed as follows (see Fig. 1):

$$\alpha(\tilde{x}, \tilde{y}) \to \alpha(v_m, v_{m+1}), \qquad (\tilde{x}, \tilde{y}) \in M_4, \qquad (v_m, v_{m+1}) \in M_3$$

It is clear from the construction that when $\alpha(x, y)$ are different, the $\alpha(v_m, v_{m+1})$ corresponding to them are also different. The proof of Theorem 1 is complete.

Remark. We used just the condition $J(x-y) = J(T(x, y) - x)$, $x, y \in Z^3$, in the proof of Theorem 1. Hence Theorem 1 holds for some non-translation-invariant potentials $J(x, y)$ as well. One can see from the proof that Theorem 1 holds for any dimension of the lattice Z^ν.

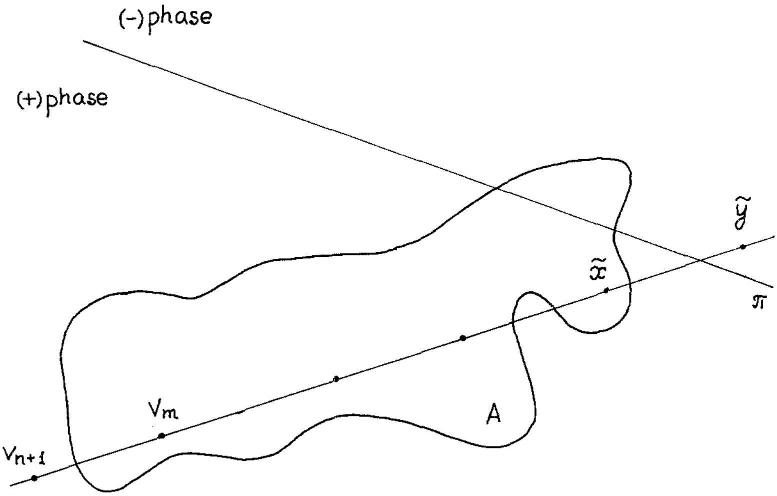

Fig. 1

3. THE STRUCTURE OF AN INTERFACE

Let us introduce several necessary definitions. A rational plane π will be assumed to be fixed. The two-dimensional lattice obtained as the intersection $\pi \cap Z^3$ will be called a derived lattice Z^2_π. The case where π is not rational is discussed in Section 8. Let $\{\mathbf{a}_1, \mathbf{a}_2\}$ be an arbitrary basis of the lattice Z^2_π such that $\mathbf{a}_1$ and $\mathbf{a}_2$ issue from a point x and $|\mathbf{a}_1| + |\mathbf{a}_2|$ is minimum. Let $\mathscr{P}$ be the family of all the planes parallel to π and containing at least one point of Z^3. The distance between two nearest different planes of $\mathscr{P}$ will be denoted by $\tilde{a}_3$:

$$\tilde{a}_3 = \min_{y \in Z^3,\, y \notin \pi} \rho(x, y)$$

The prism $V_\pi(x)$ with its center at a point $x \in Z^3$ having the linear sizes $a_1 = |\mathbf{a}_1|$, $a_2 = |\mathbf{a}_2|$, and $\tilde{a}_3$ will be called the minimal prism (MP) provided the edges of lengths a_1 and a_2 are parallel to $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, and the edge of length $\hat{a}_3$ is perpendicular to the plane π (see Fig. 2).

Let $V = \{V_\pi(x),\ x \in Z^3 \backslash x_0\}$, where $x_0 \in Z^3$ is fixed. For geometrical reasons, it is obvious that for x_0 fixed there are in general exactly 12 points x such that $V_\pi(x) \cap V_\pi(x_0) \neq \varnothing$ (for some special planes the number of neighboring MPs equals six or eight).

The distances from the point x_0 to the centers of these 12 MPs are determined by six numbers, two of which are a_1 and a_2. Let a_3, a_4, a_5, and

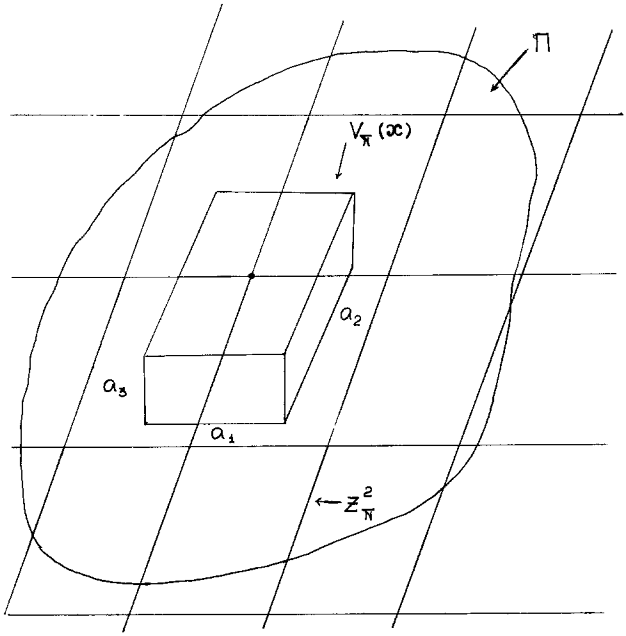

Fig. 2

a_6 denote the other four parameters. The centers of these 12 MPs are called π-nearest neighbors of a point $x_0 \in Z^3$. The faces of MP $V_\pi(x_0)$ parallel to π are called horizontal; the other four faces are called vertical. Therefore, each of two horizontal faces of $V_\pi(x_0)$ is divided into four parts, each the intersection $V_\pi(x_0)$ and $V_\pi(x)$, where x denotes a π-nearest neighbor of the point x_0. The MP $V_\pi(x_0)$ may be considered as a MP with 12 faces. Thus, for each of 12 faces of $V_\pi(x_0)$ we have the corresponding pair of π-nearest points (the distances between which are a_i, $i = 1,..., 6$), which determines the weight $J(h)$ of a face h: $J(h) = J(a_i)$. The set of all faces is divided naturally into six classes according to their weights. Let J_π denote $\min(J(a_1), J(a_2))$.

Definition 1. The boundary of the configuration $\varphi(x)$ is defined as the set

$$B^\pi(\varphi(x)) = \left[\bigcup_{\varphi(x)=1} V_\pi(x) \right] \cap \left[\bigcup_{\varphi(x)=-1} V_\pi(x) \right] \tag{10}$$

Consider a configuration $\tilde{\varphi}_\pi(x)$, which is an arbitrary finite perturbation of the ground state $\varphi_\pi(x)$ [see (3)]. The only infinite connected component of the boundary Γ^π of the configuration $\tilde{\varphi}_\pi(x)$ is called an interface and is denoted by Δ^π. Let $B^{\pi,\text{ver}}$ denote the set of all vertical faces belonging to the set B^π.

The following simple lemma gives some reasons for expecting Theorem 2 to hold.

Lemma 1. For any finite perturbation $\tilde{\varphi}_\pi(x)$ of the ground state $\varphi_\pi(x)$ there exists a positive constant c_π depending on the plane π only, such that

$$H(\tilde{\varphi}_\pi(x)) - H(\varphi_\pi(x)) \geqslant c_\pi \, |B^{\pi,\text{ver}}| \tag{11}$$

where $|B^{\pi,\text{ver}}|$ is the total area of all vertical faces.

Proof. Let $\tilde{\varphi}_\pi(x)$ be an arbitrary perturbation of the configuration $\varphi_\pi(x)$. It follows from the proof of Theorem 1 that

$$H(\tilde{\varphi}_\pi(x)) - H(\varphi_\pi(x)) = \sum_{(x,\,y) \in M} \alpha(x, y)$$

where M is a subset of the set of all pairs (x, y), $x, y \in Z^3$, and $\alpha(x, y) > 0$, $(x, y) \in M$. It follows from the definition of B^π [see (10)], $\varphi_\pi(x)$, and $B^{\pi,\text{ver}}$ that if points x and y are separated by a vertical face h, then $\tilde{\varphi}_\pi(x)\, \tilde{\varphi}_\pi(y) = -1$, $\varphi_\pi(x)\, \varphi_\pi(y) = 1$, and $\alpha(x, y) = J(x-y)$. Thus,

$$H(\tilde{\varphi}_\pi(x)) - H(\varphi_\pi(x)) \geqslant c_\pi \, |B^{\pi,\text{ver}}|$$

where

$$c_\pi = \min\left(\frac{2J(a_1)}{a_1 \tilde{a}_3}, \frac{2J(a_2)}{a_2 \tilde{a}_3}\right)$$

The proof of Lemma 1 follows.

The inequality (11) can be slightly improved by including nonvertical faces on the rhs. But we shall not consider this in detail here, since it is necessary to define walls and ceilings (see ref. 3) in order to obtain a more exact inequality.

For the following calculations, let us introduce an auxiliary cubic sublattice $\tilde{Z}^3 = Z^3(R)$ of the lattice with the spacing R and with one of its coordinate axes perpendicular to the plane π. It is easy to see that there exist values R for which $\tilde{Z}^3(R)$ exists. The way of choosing R for our calculations will be given below. For further use we define R_π to be the smallest such possible value. Let A denote the set of all possible cubes A_t (the point t is the center of the cube A_t) with vertices from $\tilde{Z}^3$ and the lengths of edges equal to R. Introduce quadrilateral parallelepipeds

$$W_{L,M} = \{x \in Z^3,\ -M \leqslant \tilde{x}_1(x) \leqslant M,\ -L \leqslant \tilde{x}_2(x) \leqslant L,\ -L \leqslant \tilde{x}_3(x) \leqslant L\}$$

where $\tilde{x}_i(x)$, $i = 1, 2, 3$, are the coordinates of the point x with respect to the coordinate system corresponding to the lattice $\tilde{Z}^3$, the axis $\tilde{x}_1$ being perpendicular to the plane π.

Now we can define contours for the models (1). Let us fix boundary conditions $\tilde{\varphi}_{W_{L,M}}(x) = \{\varphi_\pi(x), x \in Z^3 \setminus W_{L,M}\}$ and consider an arbitrary configuration $\varphi(x)$ in $W_{L,M}$. We define a contour as the pair consisting of a contour support supp K and a function $\varphi(\text{supp } K)$ that is a restriction of the configuration $\varphi(x)$ to supp K. First define the support of a thick boundary supp B^R as the set of all cubes A_t such that for each cube A_t there exist at least two π-nearest neighboring points $x, y \in Z^3$, $x, y \in A_t$, and $\varphi(x) \neq \varphi(y)$. Two cubes A_{t_1} and A_{t_2} are called connected provided $A_{t_1} \cap A_{t_2}$ consists of at least one point $x \in \tilde{Z}^3$. The connected components of supp B^R defined in such a way are called supports of contours and are denoted as supp K.

Definition 2. The pair $K = (\text{supp } K, \varphi(\text{supp } K))$ is called a contour. The set of all contours is called a thick boundary B^R of the configuration $\varphi(x)$. The contour $K = (\text{supp } \Delta, \varphi(\text{supp } \Delta)) = \Delta$ is called the interface when supp Δ is the only infinite connected component of the set supp B^R.

4. CONSTRUCTION AND INVESTIGATION OF THE CONTOUR MODEL

Following ref. 3, we investigate the statistics of an interface Δ^π. We establish its stability, which holds because the potential is long-range. It will be necessary to consider the interface Δ instead of the interface Δ^π.

Let us introduce the Gibbs distribution on the configuration space $\varphi_{W_{L,M}}(x) = \{\varphi(x), x \in W_{L,M}\}$ corresponding to the boundary condition $\bar{\varphi}_{W_{L,M}}(x) = \{\varphi_\pi(x), x \in Z^3 \setminus W_{L,M}\}$ and Hamiltonian (1). Consider the corresponding distribution on the set of all interfaces:

$$\text{Prob}(\delta = \tilde{\Delta}) = \frac{F(\tilde{\Delta})}{\sum_\Delta F(\Delta)} \tag{12}$$

where

$$F(\Delta) = \sum_{\varphi_\Delta} \exp[-\beta H(\varphi_\Delta(x))]$$

is a statistical weight of all configurations with a fixed interface Δ; the $\sum_\Delta$ in (12) means that the sum is taken over all possible interfaces Δ.

For each interface Δ there is a unique configuration $\psi_\Delta(x)$ such that $B^R(\psi_\Delta(x))=\Delta$, i.e., the boundary is exactly Δ. The set of points $x\in Z^3\backslash\operatorname{supp}\Delta$ such that $\psi_\Delta(x)=1$ $[\psi_\Delta(x)=-1]$ will be called points of the $+1$ phase (-1 phase) and will be denoted by S_Δ^+ (S_Δ^-). Let Δ be fixed. The set of points $x\in Z^3\backslash\operatorname{supp}\Delta$ such that $x\in A_t$ and $A_t\cap\operatorname{supp}\Delta$ is not empty (i.e., it consists of a face, edge, or point) is called the set of near-boundary points $\bar{S}_\Delta$. Geometrically, it is clear that the support of the interface of the configuration $\varphi(x)$ is $\operatorname{supp}\Delta$ iff $\varphi(x)=1$ $[\varphi(x)=-1]$ for the points of the $+1$ phase $[-1$ phase$]$ belonging to $\bar{S}_\Delta$. Thus, there is a bijection between the configurations

$$\varphi(x)=\{\varphi(x),\ x\in S_\Delta=W_{L,M}\backslash(\bar{S}_\Delta\cup\operatorname{supp}\Delta)\}$$

and the configurations $\varphi'(x)=\varphi'_{W_{L,M}}(x)$ such that $\Delta(\varphi'(x))=\Delta$ (see ref. 3).

According to above remarks, the formula (12) may be written in a more convenient way:

$$\operatorname{Prob}(\delta=\tilde{\Delta})=\frac{\exp[-\beta\kappa(\tilde{\Delta})]\ \Xi(S_{\tilde{\Delta}})}{\sum_\Delta\exp[-\beta\kappa(\Delta)]\ \Xi(S_\Delta)} \tag{13}$$

where $\kappa(\Delta)=H(\psi_\Delta(x)\,|\,\varphi_\pi(x))$ and $\Xi(S_\Delta)$ is a partition function corresponding to the Hamiltonian $H(\varphi(x)\,|\,\psi_\Delta(x))$ in the volume S_Δ under the boundary conditions

$$\bar{\varphi}(x)=\{\varphi_\pi(x),\ x\in Z^3\backslash W_{L,M};\ \varphi(\operatorname{supp}\Delta\cup\bar{S}_\Delta),\ x\in\operatorname{supp}\Delta\cup\bar{S}_\Delta\}$$

To investigate the formula (13) it is necessary to study the properties of the partition function $\Xi(S_\Delta)$ depending on the geometric shape of S_Δ and the boundary condition $\bar{\varphi}(x)$. It will be proven below that this dependence is rather weak.

More exactly, in this section we prove the following result:

Lemma 2. For all β large enough there exists a function $g_\beta(x,V)$, $x\in Z^3$, $V\subset Z^3$, such that

$$\ln\Xi(S_\Delta)=\sum_{x\in S_\Delta}g_\beta(x,S_\Delta) \tag{14}$$

where $g(x,V)$ has the following properties: let $d(x_1,E_1,x_2,E_2)$ be an upper bound of the set

$$[d\colon d\geqslant 0,\ (E_1-x_1)\cap\{x\in Z^3,\ |x|\leqslant d\}=(E_2-x_2)\cap\{x\in Z^3,\ |x|\leqslant d\}] \tag{15}$$

Then

$$|g_\beta(x_1,E_1)-g_\beta(x_2,E_2)|<T\exp[-\alpha d(x_1,E_1,x_2,E_2)] \tag{16}$$

for some T and α independent of β.

The method of surgeries of Gibbs distributions developed by Dobrushin is used in ref. 7 to establish the representation (14) for the Ising ferromagnet partition function with pure boundary condition. Generally speaking, the formula (14) follows from a cluster representation for the partition function.[8]

Now we explain the main idea of obtaining the decomposition (14). The routine definition of contours leads to a contour model with interaction because the potential (1) is long-range. Below we define clusters D with weights $\gamma(D)$ that will not interact, and for them the representation

$$\Xi(S_\Delta) = \sum \gamma(D_1) \cdots \gamma(D_m)$$

holds, where the sum is over the set of compatible clusters. Geometrically, these clusters D can be imagined as sets of usual contours K connected by the edges of the lattice Z^3. Two contours can be connected by several edges and the whole set has to be connected (i.e., the set cannot be divided into two parts such that any two contours of different parts are not connected). The weight $\gamma(D)$ is approximately the same as the product of the weights of the contours belonging to it. We can define correlation function of clusters and then write the Kirkwood–Salzburg equation. To be able to solve this equation it is sufficient to prove an estimate (29) from Lemma 4 which shows that the probability of a contour is small.

The subsequent calculations use the contour model method (contour models are defined below). $B^R(\varphi)$ is the set of all contours of the configuration $\varphi(x)$: $B^R(\varphi) = \{K_0 = \Delta(\varphi), K_1,..., K_n\}$. The union of all finite, connected components of the set $Z^3 \backslash \operatorname{supp} K_i$ is called the interior of the contour K_i and denoted by $\operatorname{Int} K_i$. The only infinite, connected component of the set $Z^3 \backslash \operatorname{supp} K_i$ is called the exterior of a contour and is denoted $\operatorname{Ext} K_i$. The contour K_j is external provided $\operatorname{supp} K_j \in \operatorname{Ext} K_i$, $i = 1,..., n$; $i \neq j$.[1] Geometrically it is obvious that for each contour there exists a configuration $\psi_i(x)$ such that $B^R(\psi_i(x)) = K_i$, i.e., the boundary of the configuration $\psi_i(x)$ consists of the contour K_i only. For each contour K_i its weight will be calculated by the following formulas:

$$\gamma(K_i) = \exp\left\{\frac{\beta}{2} \sum_{x, y \in Z^3} J(x-y)[\psi_i(x)\, \psi_i(y) - 1]\right\}, \qquad i = 1,..., n \tag{17a}$$

$$\gamma(K_0) = \exp[-\beta \kappa(\Delta)] \tag{17b}$$

The contour model corresponds to the formulas (17a), (17b); it is an interacting contour model because the potential is long-range. The notion of a contour should be updated to reduce the situation to one investigated earlier.[1]

The contour K_l interacts with the contour K_m through pairs (x, y) such that $x \in U_{K_l}$ and $y \in U_{K_m}$, where

$$U_{K_l} = \Big\{ x\colon x \in \operatorname{supp} K_l \cup \operatorname{Int} K_l,\quad x \notin \bigcup_{i\colon \operatorname{supp} K_i \subset \operatorname{Int} K_l} (\operatorname{supp} K_i \cup \operatorname{Int} K_i) \Big\} \tag{18}$$

$$U_{K_m} = \Big\{ x\colon x \in \operatorname{supp} K_m \cup \operatorname{Int} K_m,\quad x \notin \bigcup_{i\colon \operatorname{supp} K_i \subset \operatorname{Int} K_m} (\operatorname{supp} K_i \cup \operatorname{Int} K_i) \Big\} \tag{19}$$

and $f(x, y) \neq 0$.

The value of the interaction

$$f(x, y) = -\beta J(x-y)[\varphi(x)\,\varphi(y) - \psi_\Delta(x)\,\psi_\Delta(y) - \psi_{K_i}(x)\,\psi_{K_i}(y) - \psi_{K_j}(x)\,\psi_{K_j}(y) + 2] \tag{20}$$

Note that $|f(x, y)| < 6\beta J(x-y)$.

The following equation follows from the formulas (17a), (17b), and (20):

$$\exp[-\beta H(\varphi(x) \mid \varphi_\pi(x))] = \prod_{i=0}^{n} \exp[G(K_0, K_1,\dots, K_n)]\,\gamma(K_i) \tag{21}$$

where the multiplier $\exp[G(K_0, K_1,\dots, K_n)]$ corresponds to interaction between contours and with the boundary condition $\bar{\varphi}(x)$ [see (13)]:

$$\begin{aligned} G(K_0, K_1,\dots, K_n) &= -\beta \sum_{\substack{(x, y)\colon x \in U_{K_i}\\ y \in U_{K_j}, i \neq j}} J(x-y)[\varphi(x)\,\varphi(y) - \psi_\Delta(x)\,\psi_\Delta(y) \\ &\quad - \psi_{K_i}(x)\,\psi_{K_i}(y) - \psi_{K_j}(x)\,\psi_{K_j}(y) + 2] \\ &\quad - \beta \sum_{i=1}^{n} \sum_{\substack{(x, y)\colon x \in U_{K_i}\\ y \in \bigcap_{j=1}^{n} \operatorname{Ext} K_j}} J(x-y)[\varphi(x)\,\varphi(y) - \psi_\Delta(x)\,\psi_\Delta(y) + 1] \end{aligned} \tag{22}$$

The set of all pairs in the double sum (22) will be denoted by G. Write (21) as follows:

$$\exp[-\beta H(\varphi(x) \mid \varphi_\pi(x))] = \prod_{i=0}^{n} \gamma(K_i) \prod_{(x, y) \in G} \{1 + \exp[f(x, y)] - 1\} \tag{23}$$

From (23) we obtain

$$\exp[-\beta H(\varphi(x) \mid \varphi_\pi(x))] = \sum_{G' \subset G} \prod_{i=0}^{n} \gamma(K_i) \prod_{(x,y) \in G', f(x,y) \neq 0} g(x, y) \tag{24}$$

where the summation is taken over all subsets G' (including the empty set) of the set G, and $g(x, y) = \exp[f(x, y)] - 1$.

Consider an arbitrary term of the sum (24), which corresponds to the subset $G' \subset G$. Let the bond $(x, y) \in G'$. Consider the set $\mathscr{K}$ of all contours such that for each contour $K \subset \mathscr{K}$, $|(\text{supp } K \cup \text{Int } K) \cap \{x, y\}| = 1$. We call any two contours from $\mathscr{K}(x, y)$ connected. The set of contours $\tilde{\mathscr{K}}$ is called G' connected if for any two contours K_p and $K_q \subset \tilde{\mathscr{K}}$ there exists a collection $\{K_1 = K_p, K_2, ..., K_{n-1}, K_n = K_q\}$ such that any two contours K_i and K_{i+1}, $i = 1, ..., n-1$, are connected by some bond $(x, y) \in G'$.

Definition 3. The pair $D = [\{K_i, i = 1, ..., r\}, G']$, where G' is some set of bonds, is called a cluster provided there exists a configuration $\varphi(x)$ such that $K_i \in B^R(\varphi(x))$, $i = 1, ..., r$ (see Definition 2), $G' \subset G$ [see (24)], and the set $\{K_i, i = 1, ..., r\}$ is G' connected. The weight of a cluster D is defined by the formula

$$\gamma(D) = \prod_{i=1}^{r} \gamma(K_i) \prod_{(x,y) \in G'} g(x, y) \tag{25}$$

Two clusters D_1 and D_2 are called compatible provided any two contours K_1 and K_2 belonging to D_1 and D_2, respectively, are compatible and not connected. A set of clusters is called compatible provided any two clusters of it are compatible.

Lemma 3. Let the following boundary conditions be fixed:

$$\bar{\varphi}(x) = \{\varphi_\pi(x), x \in Z^3 \setminus W_{L,M}; \varphi(\text{supp } \Delta \cup \bar{S}_\Delta), x \in \text{supp } \Delta \cup \bar{S}_\Delta\}$$

If $\{D_1, ..., D_m\}$ is a compatible set of clusters and $\bigcup_{i=1}^{m} \text{supp } D_i \subset W_{L,M}$, then there exists a configuration

$$\varphi_{W_{L,M} \setminus \text{supp } \Delta \cup \bar{S}_\Delta}(x)$$

which contains this set of clusters. For each configuration $\varphi(x)$ we have

$$\exp[-\beta H(\varphi(x) \mid \varphi_\pi(x))] = \sum_{G' \subset G} \prod \gamma(D_i)$$

where the clusters D_i are completely determined by the set G'. The partition function is

$$\Xi(S_\Delta) = \sum \gamma(D_1) \cdots \gamma(D_m)$$

where the summation is taken over all nonordered compatible collections of clusters.

The proof of Lemma 3 follows immediately from the definitions.

We will write $\text{supp}\, K \subset \text{supp}\, D$, $D = [\{K_i, i = 1,..., r\}, G']$ provided $\text{supp}\, K = \text{supp}\, K_i$ for some i, $1 \leqslant i \leqslant r$.

Lemma 3 shows that we come to noninteracting clusters from interacting contours. Figure 3a shows contours of a configuration $\varphi(x)$ and Fig. 3b shows (one of many) clusters corresponding to the contours K_0, K_1, K_2.

For an investigation of properties of the partition function $\Xi(S_\Delta)$ we define a contour model, i.e., a family of probability distributions on the set of superboundaries formed by compatible collections of clusters D.

Definition 4. A superboundary is a finite or countable set $\{D_1,..., D_m,...\}$ of the set of all clusters $\mathscr{D}_0$ such that the collection $\partial = \{D_1,..., D_m,...\}$ is compatible. Let $\mathscr{D}$ denote the set of all superboundaries, $[\partial]$ the set of all clusters that are not compatible with at least one cluster $D_i \in \partial$, and $|\partial|$ the number of clusters contained in the superboundary ∂. The characteristic function χ_V is defined as follows:

$$\chi_V(\partial) = \begin{cases} 1 & \text{supp}\, \partial \subset V \\ 0 & \text{supp}\, \partial \not\subset V \end{cases}$$

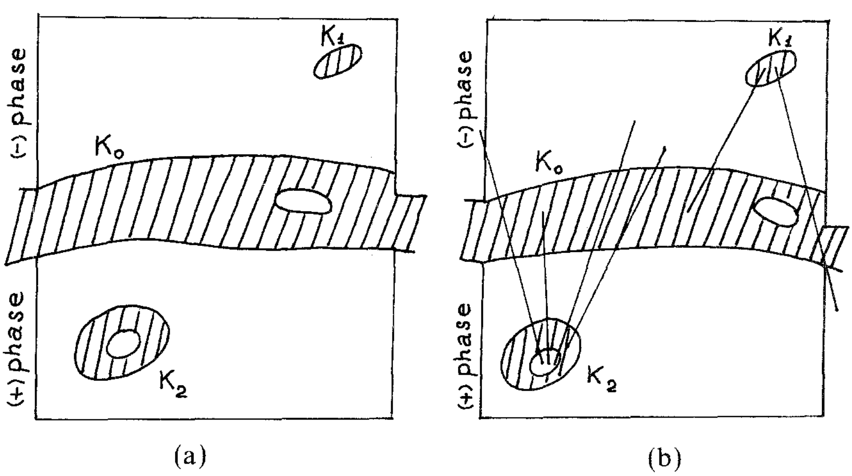

Fig. 3

The empty superboundary is supposed to belong to $\mathscr{D}$. The formula (25) determines a contour model in a standard way[1]

$$P_V(\partial) = \chi_V(\partial)\, Z^{-1}(V \mid \gamma) \prod_{D_i \in \partial} \gamma(D_i)$$

where

$$Z(V \mid \gamma) = \sum_{\partial:\, \text{supp}\, \partial \subset V} \prod_{D_i \in \partial} \gamma(D_i)$$

A correlation function is the probability that ∂ is a subset of the superboundary:

$$\rho_V(\partial) = \sum_{\tilde{\partial} \in \mathscr{D}:\, \partial \subset \tilde{\partial}} P_V(\tilde{\partial}) \tag{26}$$

The correlation function satisfies the Kirkwood–Salzburg equation[1]

$$\rho_V(\partial) = \chi_V(\partial)\, \gamma(\partial) \left[1 + \sum_{|\partial'|=1}^{|[\partial]|} (-1)^{|\partial'|} \sum_{\partial' \subset [\partial]} \rho_V(\partial') \right] \tag{27}$$

The general problem consists in investigating the behavior of ρ_V as $V \uparrow Z^3$ in the sense of Van Hove. This problem was solved[1] under the assumption that the model is given by a τ-functional, $-\ln |\gamma(\partial)| > \tau\beta\, |\partial|$, with $\tau\beta$ large enough.

Earlier E. I. Dinaburg and Ya. G. Sinai used an improved definition of a contour for studying a Potts model like the one given in Definition 3, in order to estimate an interaction. A similar approach was proposed by Bricmont *et al.*[10] It is worthwhile mentioning the papers of Mazel,[11] where some steps toward the elimination of the interaction between contours were taken also.

It follows from ref. 9 that to investigate Eq. (27) by the Minlos–Sinai method[1,12] is sufficient to establish the following lemma.

Lemma 4. For any large enough β there exists $R = R(\beta)$ such that for any supp K

$$F(\text{supp}\, K) = \sum_{D:\, \text{supp}\, K \subset \text{supp}\, D} |\gamma(D)| < \exp(-\beta\tau n) \tag{28}$$

where $\tau = 3$ and the support supp K consists of n cubes A_t.

Before proving this lemma we explain the main idea of our proof. The support supp K consists of cubes A_t with side R. The energy loss in each cube A_t is greater than or equal to 6β, and the number of contours with fixed support consisting of n cubes A_t equals 2^{nR^3}, so a rough calculation of the weight of a contour gives $\sim\exp(-6\beta n + R^3 \ln 2n)$. We have to calculate the weight of a contour in a more careful way and obtain a more exact estimate (29) of the weight of a contour.

The main point of our proof is the choice of the constant R: while R is increasing, the weight of our contour is also increasing. On the other hand, the weight of a bond $|g(x, y)|$ is decreasing, hence the "influence" of the "entropy" of all the clusters containing a given contour K is decreasing. Taking into account this circumstance, we choose R an optimal way. Thus, the "influence" of "entropy" is controlled by $\exp(-b\beta n)$ and the exponential estimation (28) remains true. The idea of chosing R in this way is a combined high-temperature, low-temperature expansion. Thus, it is sufficient to take the sum over contours that are nearest neighbors to contour K [see (33), (34)]. The final system of inequalities (i)–(vi) [after Eq. (38)] has a solution provided the potential $J(x)$ has certain decay properties. We take $J(x) = x^{-\alpha}$, where $\alpha > 9$, though it is possible to consider other potentials.

Proof. Statistical weights of contours are defined by the formulas (17a), (17b). First let us estimate the statistical weight of the contour support supp K defined by the formula

$$\bar{\gamma}(\text{supp } K) = \sum_{K_i:\, \text{supp } K_i = \text{supp } K} \gamma(K_i)$$

Let the support supp K consist of n cubes A_t. Clearly, there is at least one spin flip in each of them and the loss of energy is not less than 6β. The number of all possible locations of unit exitation in one cube is equal to R^3, which gives R^{3n} in the support supp K. Fixing the points of flips in each cube A_t, we find an upper bound on the partition function as follows: if we take into account interactions of nearest neighboring points x and y only, then the sum of statistical weights of all Ising contours passing through a fixed face will be less than $\sum_{k=1}^{\infty} \exp(-2\beta k)\, s^k$, where s is a constant (s^k is an upper bound on the quantity of Ising contours, consisting of k faces and going across a fixed face; see below Lemma 7). Moreover, note that Ising contours may pass through each of the faces of supp K, the number of faces being less than $6nR^3$. Hence

$$\bar{\gamma}(\text{supp } K) \leqslant \exp(-6n\beta)\, R^{3n} \left[1 + \sum_{k=1}^{\infty} \exp(-2\beta k)\, s^k \right]^{6nR^3} \tag{29}$$

Suppose $\beta > \ln[(2s)^{1/2}]$; then the sum in the third factor of (29) is estimated by a geometrical series with a denominator less than 0.5:

$$\bar{\gamma}(\text{supp } K) \leqslant \exp\left\{-n\left[6\beta - 3\ln R - 6R^3 \ln\left(1 + \frac{\exp(-2\beta + \ln s)}{1 - \exp(-2\beta + \ln s)}\right)\right]\right\}$$

or

$$\bar{\gamma}(\text{supp } K) \leqslant \exp\{-n[6\beta - 3\ln R - 6R^3 \exp(-2\beta + \ln 2s)]\} \qquad (30)$$

Now we estimate from above the sum (26) of statistical weights of all supports passing through a fixed cube A_t:

$$\begin{aligned} Q &= \sum_{K:\, A \subset \text{supp } K} \bar{\gamma}(\text{supp } K) \\ &\leqslant \sum_{k=1}^{\infty} \bar{\gamma}(A^k)\, s^k \\ &\leqslant \exp[-6\beta + 3\ln R + 6R^3 \exp(-2\beta + \ln 2s) + \ln 2s] \end{aligned} \qquad (31)$$

Here A^k is any connected set, consisting of k cubes A_t; the geometrical series $\sum_{k=1}^{\infty} \bar{\gamma}(A^k)\, s^k$ is summed under the assumption that the denominator is less than 0.5:

$$6\beta - 3\ln R - 6R^3 \exp(-2\beta + \ln 2s) > \ln 2s \qquad (32)$$

Let D denote an arbitrary cluster containing K: $K \subset D$. We shall say that a contour $\bar{K} \subset D$ is a neighbor of the first order of a contour K in a cluster D and write $\bar{K} \leftrightarrow K$ provided $\bar{K}$ and K are joint (see Definition 3). A contour $\bar{\bar{K}}$ is called a neighbor of the qth order for a contour K, provided $K \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_{q-1} \leftrightarrow \bar{\bar{K}}$ and there are no such diagrams with fewer arrows. Therefore, with a contour $K \subset D$ fixed, all the other contours of a given cluster D are divided into nonintersecting classes indicated by the integers 1,..., p of contours that are neighbors of the qth order for the contour K, $q = 1,..., p(D, K)$. The number $p(D, K)$ is called the order of the cluster D (the order of a cluster containing a unique contour K is zero). The contours that are neighbors of qth order of a fixed contour K will be denoted by $K_q(K)$.

Besides the weight $\bar{\gamma}(\text{supp } K)$ of each support supp K we introduce the new weight $\bar{\bar{\gamma}}_b(\text{supp } K) = \bar{\gamma}(\text{supp } K) \exp(\beta n b)$ provided supp K consists of n

cubes A_t: $|\text{supp } K| = n$. Then, analogously to formula (31), we can write for such a weight

$$Q^b = \sum_{K:\, A \subset \text{supp } K} \bar{\bar{\gamma}}_b(\text{supp } K)$$
$$\leqslant \exp[-6\beta + 3 \ln R + 6R^3 \exp(-2\beta + \ln 2s) + \ln 2s + b\beta] \quad (31')$$

We can write the condition (32) as follows:

$$6\beta - 3 \ln R - 6R^3 \exp(-2\beta + \ln 2s) - b\beta > \ln 2s \quad (32')$$

Suppose that we have proved the following inequality for a supp K_i:

$$\sum_{D:\, \text{supp } D = \{\text{supp } K_i, \text{supp } K_{i+1}\}} \bar{\bar{\gamma}}_\beta(D) \leqslant \bar{\gamma}(\text{supp } K_i) \exp(nb\beta) \quad (33)$$

where the sum is taken over all the clusters consisting of a given supp K_i and the supp K_{i+1}, which are neighbors of $(i+1)$th order of the contour K. The weight of a cluster $\bar{\bar{\gamma}}_\beta(D)$ is the calculated according to Definition 3:

$$\bar{\bar{\gamma}}_\beta(D) = \gamma(K_i)\, \bar{\bar{\gamma}}_b(K_{i+1}) \exp[G(K_0, K_1, ..., K_n)]$$

where the weight of the contour K_{i+1} is chosen as $\bar{\bar{\gamma}}_b(K_{i+1}) = \gamma(K_{i+1})\, e^{nb\beta}$.

As we can see, $\bar{\bar{\gamma}}_b(D) = \gamma(D)\, e^{nb\beta}$.

Now by induction on the order of a cluster it is not difficult to obtain the inequality

$$F(\text{supp } K) \leqslant \bar{\gamma}(\text{supp } K)\, e^{nb\beta} = \bar{\bar{\gamma}}_b(\text{supp } K)$$

from the inequality (33).

It can be shown that

$$\sum_{D:\, \text{supp } D = \{\text{supp } K_i, \text{supp } K_{i+1}\}} \gamma(D)$$
$$\leqslant \bar{\gamma}(\text{supp } K_i) \prod_{x \in U_{K_i}} \left[1 + \sum_{g(x,y) \in G(x,y)} |g(x, y)|\, M_1 M_2\right]$$
$$\equiv \bar{\gamma}(\text{supp } K_i) \cdot Z \quad (34)$$

The multipliers M_1 and M_2 are associated with the contours separating the point x and y from infinity, respectively. Assume that

$$Q^b < 1, \qquad Q^b_m = \sum_{K:\, x \in \text{supp } K,\, |\text{supp } K| \geqslant m} \bar{\bar{\gamma}}_\beta(\text{supp } K) < e^{-2\beta m} \quad (35)$$

It can be shown that

$$M_1 < \prod_{k=1}^{\infty} (1 + Q_{8k}^b)$$

Finally,

$$\sum_{k=1}^{\infty} \ln(1 + Q_{8k}) < \sum_{k=1}^{\infty} Q_{8k} < \exp(-14\beta)$$

and $M_1 < \exp[\exp(-14\beta)]$. Similarly, $M_2 \leqslant (1 + Q^b)\, M_1$.

To each "bond" $g(x, y)$ from (34) such that $x \in \operatorname{Int} K$ we assign the point $z \in \operatorname{supp} K$ such that z is an integer point closest to the segment $[x, y]$ (if there are several of them, then we choose the first one under some numeration of the points of Z^3). Bearing in mind that for each point $z \in \operatorname{supp} K$ there exist no more than $\frac{4}{3}\pi r^3$ "bonds" $g(x, y)$ with the lengths $r = |x - y|$ assigned to it, we estimate the right-hand side of (34) $(M_1 \cdot M_2 < 2e^2 < 18)$:

$$\begin{aligned} Z &\leqslant \prod_{\tilde{x} \in \operatorname{supp} K} \left[1 + \frac{18 \cdot 4}{3} \pi \sum_{g(\tilde{x}, y) \in G(x, y)} r^3 \, |g(x, y)| \right] \\ &\leqslant \left[1 + 24\pi \sum_{g(x, y) \in G_{\tilde{x}}^R} r^3 g(x, y) \right]^{nR^3} \equiv (1 + Y)^{nR^3} \end{aligned} \tag{36}$$

where

$$G_{\tilde{x}}^R = \{ g(\tilde{x}, y) \colon |\tilde{x} - y| > R \}$$

Finally, we see from the expressions (30), (31), (34), and (36) that for (33) to hold it is sufficient that

$$\exp(-nb\beta) \exp[nR^3 \ln(1 + Y)] \leqslant \exp[-n(b\beta - R^3 Y)] < 1 \tag{37}$$

But

$$\exp[-n(b\beta - R^3 Y)] \leqslant \exp[-n(b\beta - \beta \cdot 10^7 R^3)] \tag{38}$$

Explanation of (38): (a) $|g(x, y)| < 12\beta J(r)$ when $6\beta J(r) < 1$. (b) The points $x \in Z^3$ are calculated in the following way: $r \in Z^1$ fixed, there are just $6 \cdot 4r^2$ points on the surface of the cube with its center at the origin and side length $2r$. Their distance to the coordinate origin is between r and $3^{1/2} r$.

Finally, we obtain from (30)–(38) that for (28) to hold it is sufficient to choose β, $R(\beta)$, and b in such a way that the following system can be solved:

(i) $b > 10^7 R^3 \sum_{r=R+1}^{\infty} r^5 J(r)$

(ii) $6\beta J(r) < 1$ when $r > R$

(iii) $\beta > \ln[(2s)^{1/2}]$

(iv) Condition (35)

(v) Condition (32′)

(vi) $6\beta - 3 \ln R - 6R^3 \exp(-2\beta + \ln 2s) - b_\beta > \tau\beta$

Condition (i) shows that the "stock" b must control the influence of interaction.

The Solution of the System (i)–(vi). First, (v) follows from (iv). Below the solution is sought for potentials $J(r) = r^{-\alpha}$, $\alpha > 8$. We are not going to consider the temperature interval as large as possible. We show just a variant of the solution. Suppose $\beta > 3\ln(2sR)$. Then (iii) holds automatically and

$$\begin{aligned} Q_m^b &\leqslant \exp\{-m(6\beta - 3\ln R - 6R^3 \exp(-2\beta + \ln 2s) - \ln 2s - b\beta)\} \\ &\leqslant \exp\{-m(2\beta + (\beta - 3\ln R) + (\beta - 6R^3 \exp(-2\beta + \ln 2s)) + (\beta - b\beta)\} \\ &\leqslant \exp(-2\beta m) \end{aligned} \tag{39}$$

where $b = 1$. It is proved exactly as in (39) that (vi) is true with $\tau = 3$. Then we choose

$$R = \left[\frac{\exp(\beta/3)}{2s}\right]_\pi$$

Here $[c]_\pi$ denotes the greatest integer less than c and divisible by R_π.

We see that (i) and (ii) hold,

$$1 > 10^7 R^3 \sum_{r=R+1}^{\infty} r^{5-\alpha} \tag{40}$$

$$6\beta R^{-\alpha} < 1 \tag{41}$$

Indeed, substitute $e^{\beta/3}/2s - R_\pi$ instead of R and sum the series in (40),

$$(e^{\beta/3}/2s - R_\pi)^{\alpha-8} > 10^7 \tag{42}$$

and

$$(e^{\beta/3}/2s - R_\pi) > (6\beta)^{1/\alpha} \tag{43}$$

Obviously, for α fixed there exists β_1 such that (42) and (43) hold when $\beta > \beta_1$, $R = (e^{\beta/3}/2s)_\pi$ and $\tau = 3$, $b = 1$. The proof is complete.

We will use the contour method of Minlos and Sinai[1,12] and consider the right-hand side of (27) as an operator in Banach space of boundary functionals. A boundary functional $\xi = \xi(\partial)$ is a real function on the set of finite superboundaries. The set of boundary functionals forms a linear space. We introduce the following norm $\|\xi\|_W$, $W \subset Z^3$, of a boundary functional:

$$\|\xi\|_W = \sup_{\partial \subset \mathscr{D}} \frac{\xi(\partial)}{\exp[a\beta\,|\text{supp}\,\partial| - F(\text{supp}\,\partial) + (ab - \tau\beta)\,\text{dist}(\partial, \bar{W})]}$$

where $a = s + b$,

$$\text{dist}(\partial, \bar{W}) = \min_{K \subset \partial} \text{dist}(\text{supp}\,K, Z^\nu \setminus W)$$

and

$$F(\text{supp}\,\partial) = \sum_{D:\ \text{supp}\,D = \text{supp}\,\partial} \gamma(D)$$

Every such norm determines the Banach space B_W of boundary functionals. The right-hand side of (27) determines an operator E on boundary functionals:

$$(E\xi)(\partial) = \gamma(\partial) \sum_{|\partial'| = 1}^{|[\partial]|} (-1)^{|\partial'|} \sum_{\partial' \subset [\partial]} \xi(\partial') \tag{44}$$

With the new definition (27) can be rewritten as

$$\xi = \chi_V \gamma + \chi_V E \chi_V \xi, \qquad V \subset Z^3 \text{ is finite} \tag{45}$$

The solutions of the equation

$$\xi = \gamma + E\xi$$

will be called correlation functions in an infinite volume, since the correlation function satisfies (45).

Using Lemma 4 and the definition of $\|\xi\|_W$, we can prove the following lemma in a way analogous to refs. 1 and 12.

Lemma 5. There exists β_1 such that $\|E\|_W < e^{-a}$ for $\beta > \beta_1$.

Further, proceeding as in ref. 1, it is possible to obtain all the results concerning correlation decay and decomposition of a free energy into a volume and a boundary part. In this way one can obtain Lemma 2.

5. STATISTICAL PROPERTIES OF AN INTERFACE

Using Lemma 2, we study the statistical properties of an interface Δ. For convenience we express the distribution of a value δ via the geometrical properties of an interface.

Let $H_{W_{L,M}}$ denote the set of all the cubes $A_t \subset W_{L,M}$. Let $\mathfrak{M}_{W_{L,M}}$ denote the set of all interfaces Δ of the configurations $\tilde{\varphi}_\pi(x)$ [recall that $\tilde{\varphi}_\pi(x)$ is a finite perturbation of $\varphi_\pi(x)$ in the volume $W_{L,M}$]; T_W denotes the set of all centers t of the cubes $A_t \subset W$.

Lemma 6. For all β large enough there exists a function $f_\beta(t, \Delta, W_{L,M})$ defined for all integer numbers L, M, L/R, M/R for all the interfaces $\Delta \in \mathfrak{M}_{W_{L,M}}$ for all the centers t of the cubes $A_t \in H_{W_{L,M}}$ such that:

(I) For some $C < \infty$, $c > 0$ independent of β, any $W = W_{L,M}$, $\tilde{W} = W_{\tilde{L},\tilde{M}}$, any $\Delta \in \mathfrak{M}_W$, $\tilde{\Delta} \in \mathfrak{M}_{\tilde{W}}$, and any centers $t, \tilde{t}$ of the cubes $A_t \in H_W$, $A_{\tilde{t}} \in H_{\tilde{W}}$, the following two inequalities hold:

$$|f_\beta(t, \Delta, W)| \leqslant C \tag{46}$$

$$|f_\beta(t, \Delta, W) - f_\beta(\tilde{t}, \tilde{\Delta}, \tilde{W})| \leqslant C \exp[-cd(t, \Delta, W, \tilde{t}, \tilde{\Delta}, \tilde{W})] \tag{47}$$

where $d(t, \Delta, W, \tilde{t}, \tilde{\Delta}, \tilde{W})$ is an upper bound of the number d such that

$$\begin{aligned} (W - t) \cap \{x \in R^3: |x| \leqslant d\} &= (\tilde{W} - \tilde{t}) \cap \{x \in R^3: |x| \leqslant d\} \\ (\operatorname{supp} \Delta - t) \cap \{x \in R^3: |x| \leqslant d\} &= (\operatorname{supp} \tilde{\Delta} - \tilde{t}) \cap \{x \in R^3: |x| \leqslant d\} \end{aligned} \tag{48}$$

(II) The probability of the interface calculated in the Gibbs distribution corresponding to the Hamiltonian (1) and boundary conditions $\varphi(x) = \{\varphi_\pi(x), x \in Z^3 \backslash W_{L,M}\}$ is equal to [see the formula (13)]

$$\operatorname{Prob}(\delta = \Delta) = (Z_{L,M})^{-1} \exp\left\{[-\beta\kappa(\Delta)] + \sum_{t \in T_W \cap \bar{S}_\Delta} f_\beta(t, \Delta, W)\right\}, \qquad \Delta \in \mathfrak{M}_W \tag{49}$$

where

$$Z_{L,M} = \sum_{\Delta \in \mathfrak{M}_W} \exp\left\{[-\beta\kappa(\Delta)] + \sum_{t \in T_W \cap \bar{S}_\Delta} f_\beta(t, \Delta, W)\right\} \tag{50}$$

Proof. Taking into account the analogy with ref. 3, Lemma 1, we present just a sketch of the proof.

Define

$$f_\beta(t, \Delta, W) = \sum_{x \in S_\Delta:\, h_x^W(\Delta) = t} [g_\beta(x, S_\Delta) - g_\beta(x, W)] - \sum_{x \in \bar{S}_\Delta:\, h_x^W(\Delta) = t} g_\beta(x, W) \tag{51}$$

where for any point x, $h_x^W(\Delta)$ denotes a center $t \in T_{L,M}$ nearest to x (if there are several such centers, then $h_t^W(\Delta)$ is the first point by the lexicographic ordering of all the centers).

It is not difficult to see that

$$\ln \Xi(S_\Delta) = \sum_{t \in T_W \cap \bar{S}_\Delta} f_\beta(t, \Delta, W) + \sum_{x \in W} g_\beta(t, W) \tag{52}$$

Thus, (49) follows from (13) and (51). We shall not check the properties (46), (47) of the function $f_\beta(t, \Delta, W)$ (see ref. 3). Lemma 6 has been proved.

Now we study the interface in an infinite parallelepiped W_L. Let $\mathfrak{M}_L$ be the union of $\mathfrak{M}_{W_{L,M}}$. Obviously, $\mathfrak{M}_L$ is the set of those interfaces from $\mathfrak{M}_{W_{L,\infty}}$ that consist of a finite number of cubes Λ_t. Lemma 6 [the inequality (47)] ensures the existence of the limit

$$f_\beta(t, \Delta, L) = \lim_{M \to \infty} f_\beta(t, \Delta, W_{L,M}) \tag{53}$$

for all $\Delta \in \mathfrak{M}_L$, L, and $t \in T_{W_L} \cap \bar{S}_\Delta$. The function $f_\beta(t, \Delta, L)$ satisfies the conditions (when β is large)

$$|f_\beta(t, \Delta, L)| < C \tag{54}$$

$$|f_\beta(t, \Delta, L) - f_\beta(\tilde{t}, \tilde{\Delta}, L)| \leqslant C \exp[-\bar{\alpha} d(t, \Delta, W_L, \tilde{t}, \tilde{\Delta}, W_L)] \tag{55}$$

where $d(..., ...)$ is defined in Lemma 6. Thus, we introduce the probability distribution on the set of the interfaces $\Delta \in \mathfrak{M}_L$:

$$\operatorname{Prob}^L(\Delta) = (Z_L)^{-1} \exp\left[-\beta\kappa(\Delta) + \sum_{t \in T_{W_L} \cap \bar{S}_\Delta} f_\beta(t, \Delta, L)\right] \tag{56}$$

where

$$Z_L = \sum_{\Delta \in \mathfrak{M}_L} \exp\left[-\beta\kappa(\Delta) + \sum_{t \in T_{W_L} \cap \bar{S}_\Delta} f_\beta(t, \Delta, L)\right] \tag{57}$$

By showing Z_L to be finite, we will show that the definitions (56), (57) are correct.

Lemma 7. The number of all connected subsets of the set of all cubes A including one fixed cube and consisting of n cubes is not greater than s^n, $s < \infty$.

Proofs of analogous lemmas are presented in many papers (see ref. 1, Chapter 2, Lemma 5, or ref. 3, Lemma 2).

Using Lemma 7, the estimates (54) and (57), and the proof of Lemma 3, we obtain

$$Z_L \leqslant \sum_{\Delta \in \mathfrak{M}_L} \exp[-\beta\kappa(\Delta) + C\,|\mathrm{supp}\,\Delta|]$$
$$\leqslant \sum_{k=1}^{\infty} s^k \exp[-k(\beta J_\pi + C)] < \infty$$

where $|\mathrm{supp}\,K|$ denotes the number of all cubes $\Delta_t \in \mathrm{supp}\,\Delta$.

The definitions of (56), (57) are correct for $\beta J_\pi > \max((C + \ln s)/J_\pi, \beta_1/J_\pi, \beta_2/J_\pi)$ or $\beta J_\pi > \mathrm{const}$. Analogously, one can prove that

$$\lim_{M \to \infty} Z_{L,M} = Z_L \tag{58}$$

The following lemma allows us to reduce the investigation of the Gibbs field in W_L to the investigation of the geometrical properties of an interface Δ.

Lemma 8. Let $\mathrm{Prob}\{\varphi(x), x \in W_L\}$ be the Gibbs distribution in W_L corresponding to the Hamiltonian (1) and the boundary conditions $\bar{\varphi}_{W_L(x)} = \{\varphi_\pi(x), x \in Z^3 \setminus W_L\}$; let Δ be the interface of the configuration $\{\varphi(x), x \in W_L\}$. Then $\Delta \in \mathfrak{M}_L$ with the probability 1, and, moreover,

$$\mathrm{Prob}(\delta = \Delta) = \mathrm{Prob}^L(\Delta), \qquad \Delta \in \mathfrak{M}_L \tag{59}$$

Proof. This follows from Lemma 6 and (53), (58), (56):

$$\mathrm{Prob}(\delta = \Delta) = \lim_{M \to \infty} (Z_{L,M})^{-1} \exp\left[-\beta\kappa(\Delta) + \sum_{t \in T_W \cap \bar{S}_\Delta} f_\beta(t, \Delta, W)\right]$$
$$= \mathrm{Prob}^L(\Delta) \tag{60}$$

Due to (56) and (57) the sum of probabilities (59) taken over all $\Delta \in \mathfrak{M}$ equals 1, and Lemma 8 follows.

6. THE GEOMETRICAL STRUCTURE OF AN INTERFACE

To study formula (59) it is necessary to establish some geometrical features of the interface Δ. The structure of an interface of an Ising ferromagnet is studied in ref. 3. Because of the analogy with ref. 3, the proofs of the geometrical facts in this section will be omitted.

The interface of the ground state $\varphi_\pi(x)$ will be called regular and denoted by $\Delta(\pi)$. By the projection of a cube A_t with center at the point $t=(\tilde{t}_1(t), \tilde{t}_2(t), \tilde{t}S1_3(t))$, where $\tilde{t}_i(t)$ are the coordinates of the point t in the coordinate system defined by lattice $\tilde{Z}^3$, we mean the cube $p(A_t)=A_{p(t)}$, where $p(t)=(-\hat{t}_0, \tilde{t}_2(t), \tilde{t}_3(t))$. Here $\hat{t}_0$ denotes the distance between the center of any cube belonging to the regular interface and the plane π.

Assume that the interface Δ is fixed. We will say that a cube A_t is a ceiling cube if there exists no other cube A_{t_1} such that $p(A_t)=p(A_{t_1})$, and the boundary (not a thick boundary) $B^\pi(\varphi(x))$ intersects the cube A_t in a horizontal face

$$h=\{x=(\tilde{x}_1(x), \tilde{x}_2(x), \tilde{x}_3(x)): \tilde{x}_1(x)=\tilde{t}_1(t)+\tilde{\alpha},$$
$$|\tilde{x}_2(x)-\tilde{t}_2(t)| \leqslant R/2, |\tilde{x}_3(t)-\tilde{t}_3(t)| \leqslant R/2\}$$

where $|\tilde{\alpha}| \leqslant R/2$ is a number determining the location of an interface with respect to the cube A_t. The last condition means that $\varphi(A_t)$ is obtained as the restriction of the ground state $\varphi_{\pi'}(x)$ on the cube A_t when π' is parallel to π. Obviously, all the cubes of a regular contour are ceiling cubes. The rest will be called wall cubes. Walls S are defined as connected components of the set of all wall cubes. Let us divide all ceiling cubes into maximal subsets consisting of connected unions of cubes having the same distance from the plane π. We call these subset ceilings T. Evidently, there exists only one ceiling containing all the cubes of the regular interface. This ceiling will be called regular.

Complications in defining walls and ceilings arise because there may be many shapes (some patterns are shown in Fig. 4) of the interface. The crosses in Fig. 4 denote the cubes that are already wall cubes independently of $\varphi(A_t)$.

Below we shall show that when β is large, the probability of the event that a wall cube is located in a given place is small. For this purpose we use the usual Peierls argument. The introduction of standard walls and wall groups is necessary because of the interaction of close and high walls.

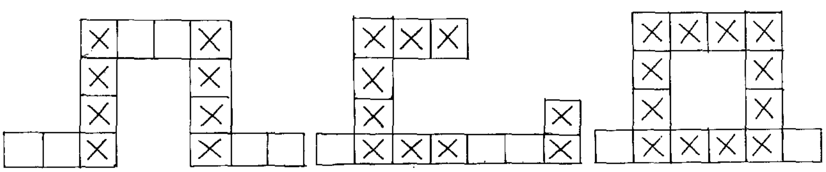

Fig. 4

By the projection $p(S)$ of a wall S we mean the union of all the cubes that are projections of the cubes of S. The projection $p(T)$ of a ceiling is defined analogously. The ceiling neighboring to the wall S, the projection of which belongs to the infinite, connected component of the complement to $p(S)$, will be called the base of S [the complement is considered with respect to the support of the regular contour of the interface $\Delta(\pi)$].

Lemma 9. Each wall has only one base.

A wall S for which there exists an interface $\tilde{\Delta} \in \mathfrak{M}_L$ such that S is the only wall for supp $\tilde{\Delta}$ is called standard. A cube belonging to the interface $\Delta(\pi)$ is called interior for the projection $p(S)$ if its interior points belong to $p(S)$ or a finite component of the complement of $p(S)$. The cubes interior for the projection $p(T)$ of a ceiling T are defined analogously. The set of the centers of interior cubes for the projection $p(S)$ of a standard wall S will be denoted by Int S. The point from Int S having the minimum number under some numeration of the centers of all cubes will be called the origin of a standard wall S. The existence of the origin for any standard wall is evident. Let T_π denote the set of all centers of the cubes belonging to supp $\Delta(\pi)$. We define the empty wall Λ_t with the origin at a point $t \in T_\pi$. The number of cubes belonging to the wall Λ_t is equal to zero; $p(\Lambda_t)$ is the empty set. Let $\{S\}_t$ denote the set of all standard walls with the origin at a point t. The set $\{S_t, t \in T_\pi\}$ (where $S_t \in \{S\}_t$) will be called an admissible collection of standard walls, provided there exists a configuration $\{\varphi(x)\}$ such that $\Delta(\varphi)$ consists of these walls. Let O_L denote the set of all admissible collections. The height of a ceiling T is the number $v = \text{dist}(A_t, \text{supp}\, \Delta(\pi)) + R$, where A_t is an arbitrary cube of the ceiling T. The shift by ϑ (where ϑ/R is integer) of set E of cubes is the set $E_v = \{A_t \colon A_t - v \in E\}$, where $\bar{\vartheta} = (\vartheta, 0, 0)$.

The shift $Q(S)$ of a wall S means its shift by $-\vartheta$, where ϑ is its base height.

Lemma 10. The shift $Q(S)$ of any wall S is a standard wall.

Let

$$\mathscr{B}_t(\Delta) = \begin{cases} Q(S) & \text{if } \; Q(S) \in \{S\}_t \\ \Lambda_t & \text{if } \; Q(S) \notin \{S\}_t \end{cases}$$

for all S belonging to supp Δ. Then $\{\mathscr{B}_t(\Delta), t \in T_\pi\} \in O_L$ and in this way a bijection is determined between the set of an interface contour $\Delta \in \mathfrak{M}_L$ and the set of admissible collections of standard walls O_L.

Lemma 11. Suppose that for two interfaces $\Delta \in \mathfrak{M}_L$ and $\tilde{\Delta} \in \mathfrak{M}_L$ the following two conditions hold: (1) $\mathscr{B}_t(\Delta) = \mathscr{B}_t(\tilde{\Delta})$, $t \in T_\pi \setminus \{t_0\}$ and (2) $\mathscr{B}_{t_0}(\Delta) \neq \mathscr{B}_{t_0}(\tilde{\Delta}) = \Lambda_{t_0}$. In addition, let $T_1, \ldots, T_l$ be the ceilings of the interface Δ that are neighboring to the wall $Q^{-1}(\mathscr{B}_{t_0}(\Delta))$, T_1 being its base, and let d_i, $i=1,\ldots,l$, be the difference between the heights of the ceilings T_1 and T_i. Finally, let I_i, $i=1,\ldots,l$, be the set of cubes such that interior points of their projections belong to a connected component of the complement to $p(\mathscr{B}_{t_0}(\Delta))$ such that $T_i \subset I_i$ and let $I=\bigcup_{i=1}^{l} I_i$. Then it is possible to establish a bijection between the cubes from supp $\Delta \cap I$ into supp $\tilde{\Delta} \cap I$. It is obtained by shifting by d_i the cube A_t when $A_t \in I_i$. Except for the cubes from supp $\tilde{\Delta} \cap I$, the interface contains only the set of cubes belonging to the base of T_1 and projected into cubes of $p(\mathscr{B}_{t_0}(\Delta))$.

7. INTERFACE SHARPNESS

Now we proceed to the final part of the proof of Theorem 2. With each standard wall we associate the number

$$\omega(S) = |S| - |\tilde{p}(S)| \tag{61}$$

where $\tilde{p}(S)$ is a maximum subset of the set $p(S)$ such that for every cube $A_{t_1} \in \tilde{p}(S)$ there exists a cube A_t such that $p(A_t) = A_{t_1}$ and $\varphi(A_t)$ equals the restriction of $\varphi_{\pi'}(x)$ for some $\pi' \parallel \pi$; $|S|$ and $|\tilde{p}(S)|$ denote the numbers of cubes belonging to S and $\tilde{p}(S)$, respectively. It follows from the definition of the wall that

$$\omega(S) \geqslant |S|/2, \qquad \omega(S) \geqslant p(S) \tag{62}$$

and, moreover, if t_1 and t_2 are the centers of cubes that are interior relative to the projection of the wall S, then

$$|t_1 - t_2| < \omega(S) - 1 \tag{63}$$

It follows that $\omega(S) > 1$ for any nonempty wall.

Consider the set of random variables $\{\eta_t, t \in T_\pi\}$ with values in $\{S\}_t$ such that

$$\operatorname{Prob}(\eta_t = \mathscr{B}_t, t \in T_\pi) = \begin{cases} \sum_{\Delta} \operatorname{Prob}^L(\Delta) & \text{if } \mathscr{B}_t = \mathscr{B}_t(\Delta), \quad t \in T_\pi \\ 0 & \text{if } \{\mathscr{B}_t, t \in T_\pi\} \notin O_L \end{cases} \tag{64}$$

The formula (64) will be fundamental for further estimations.

Let $\tilde{t} \in T_\pi$ and let $\varepsilon(\tilde{t})$ be the number of cubes A_t such that $p(A_t)$ has its center located at a point $\tilde{t}$. Standard walls S_1 and S_2 will be called neighboring provided that for some points $t_1 \in \operatorname{Int} S_1$, $t_2 \in \operatorname{Int} S_2$

$$|t_1 - t_2| < [\varepsilon(t_1)]^{1/2} + [\varepsilon(t_2)]^{1/2} \tag{65}$$

Let us call walls S_1 and S distant provided they are not neighboring. The set of standard walls N is called a group of walls if N is a collection of admissible walls and for any walls $S \in N$ and $\tilde{S} \in N$ there exist walls $S_1 = S$, $S_2 \in N,..., S_{n-1} \in N$, $S_n = \tilde{S}$, such that S_k and S_{k+1} are neighboring walls for $k = 1,..., n-1$. The origin of the wall group N is the point $t \in T_\pi$, which is minimal with respect to its number among all origins of the walls $S \in N$. Let $\bar{\Lambda}_t$ be an empty wall group with the origin located at a point t. The set of all wall groups with the same origin located at a point t is denoted by $\{N\}_t$. The collection of wall groups $\{N_t, t \in T_\pi\}$ is called admissible if the set of all the walls belonging to the wall groups $\{N_t, t \in T_\pi\}$ is admissible and any two walls $S_1 \in N_k$, $S_2 \in N_l$, $k \neq l$, are distant.

It can be derived from Lemma 10 that it is possible to establish a bijection between the set of supports of interfaces and the set of admissible collections of wall groups $\{N_t(\Delta), t \in T_\pi\}$ in such a way that the collections of standard walls $\{S_t, t \in T_\pi\}$ contain those and only those nonempty walls that belong to the groups of walls $N_t(\Delta)$. Therefore, it is useful to introduce the set of random variables $\{\xi_t, t \in T_\pi\}$ with values in $\{N\}_t$ such that [see (64)]

$$\operatorname{Prob}(\xi_t = N_t, t \in T_\pi) = \begin{cases} \sum_{\Delta} \operatorname{Prob}^L(\Delta) & \text{if } \quad N_t = N_t(\Delta), t \in T_\pi \\ 0 & \text{if the collection } \{N_t, t \in T_\pi\} \\ & \text{is not admissible} \end{cases} \tag{66}$$

Suppose that $\Delta \in \mathfrak{M}_L$, $\tilde{\Delta} \in \mathfrak{M}_L$, $N_l(\Delta) = N_l(\tilde{\Delta}) = N_l$; $\varphi(N_l(\Delta)) = \varphi(N_l(\tilde{\Delta}))$, $l \in T_\pi \setminus \{t\}$; $N_t(\Delta) = N_t \neq \Lambda_t = N_t(\tilde{\Delta})$.

Lemma 12. There exists $\tilde{\beta}$ such that for all $\beta > \tilde{\beta}/J_\pi$ for any L, any $t \in T_\pi$, and any $N_l \in \{N\}_l$, $l \in T_\pi$, the conditional probability is given by

$$\begin{aligned} &\operatorname{Prob}(\xi_t = N_t, \varphi(\xi_t) = \varphi(N_t) \mid \xi_l = N_l, \varphi(\xi_l) = \varphi(N_l), l \in T_\pi \setminus \{t\}) \\ &\quad \leqslant \exp\{-\beta[\kappa(\Delta) - \kappa(\tilde{\Delta})] + \omega(N_t) \cdot \text{const}\} \end{aligned} \tag{67}$$

if $\{N_l, l \in T_\pi \setminus \{t\}, \Lambda_t\}$ is an admissible collection of wall groups. Here $\omega(N_t) = \sum_{S \in N_t} \omega(S)$, and J_π is a constant depending on the plane π only:

Proof. Using (61), (50), and (66), one can show that

$$\begin{aligned} &\operatorname{Prob}(\xi_t = N_t, \varphi(\xi_t) = \varphi(N_t) \mid \xi_l = N_l, \varphi(\xi_l) = \varphi(N_l), l \in T_\pi \setminus \{t\}) \\ &\quad \leqslant \frac{\operatorname{Prob}^L(\Delta)}{\operatorname{Prob}^L(\tilde{\Delta})} \\ &\quad \leqslant \exp\{-\beta[\kappa(\Delta) - \kappa(\tilde{\Delta})] + \omega(N_t) \cdot \text{const}\} \\ &\qquad \times \exp\left[\sum_{t \in T_W \cap \bar{S}_\Delta} f_\beta(t, \Delta, L) - f_\beta(t, \tilde{\Delta}, L)\right] \end{aligned} \tag{68}$$

Now using (47), (60) from Lemma 6, Lemma 11, the definition of a group of walls [see (65)], and the method of ref. 3, one can complete the proof of Lemma 12 without difficulties.

Note that it follows from (67) (and from the fact that if $\{F_l, l \in T_\pi \setminus \{t\}, A_t\}$ is not an admissible collection of groups of walls, then surely $\{F_l, l \in T_\pi\}$ is not an admissible collection) that for all $N_t \in \{N\}_t$, $t \in T_\pi$,

$$\begin{aligned}\mathrm{Prob}(\xi_t = N_t, \varphi(\xi_t) = \varphi(N_t)) &\leqslant \exp\{-\beta[\kappa(\Delta) - \kappa(\tilde{\Delta})] + \omega(N_t) \cdot \mathrm{const}\} \\ &\leqslant \exp[-\tfrac{3}{2}\beta J_\pi \omega(N_t)]\end{aligned}$$

Finally, similarly to (39),

$$\sum_{\varphi(N_t)} \mathrm{Prob}(\xi_t = N_t, \varphi(\xi_t) = \varphi(N_t)) \leqslant \exp[-\beta J_\pi \omega(N_t)] \tag{69}$$

Lemma 13. The number of different groups of walls N_t such that $\omega(N_t) = k$ does not exceed $\bar{s}^k$, where $\bar{s}$ is a constant. The proof is analogous to the proof of Lemma 9 from ref. 3 and is omitted.

By a regular ceiling we mean the ceiling belonging to the support of a regular interface $\Delta(\pi)$.

Lemma 14. There exists a constant M_0 such that under the condition $\beta J_\pi > \tilde{\beta}$, for all L and for any cube $A_t \in \operatorname{supp} \Delta(\pi)$ the probability $p(L, A_t)$ that this cube does not belong to a regular ceiling is not greater than $M_0 \exp(-\frac{1}{2}\beta J_\pi)$.

Proof. It follows from the definitions, the properties of ceilings (see Section 6), and the definition of a regular ceiling that if $A_t \in \operatorname{supp} \Delta(\pi)$ does not belong to a regular ceiling, then A_t is interior for the projection of a wall of an interface Δ. In turn, if a cube A_t is interior for the projection of a wall belonging to a group $N_{\tilde{t}}$ of walls, then it follows from (63) that $|t - \tilde{t}| + 1 \leqslant \omega(N_{\tilde{t}})$.

Thus,

$$p(L, A_t) \leqslant \sum_{t \in T_\pi} \mathrm{Prob}(\omega(\xi_t) \geqslant |t - \tilde{t}| + 1) \tag{70}$$

Applying Lemma 13 and inequality (69), we obtain that when β is large enough, there exists a constant M_1 such that

$$\begin{aligned}&\mathrm{Prob}(\omega(\xi_t) \geqslant |t - \tilde{t}| + 1) \\ &\quad \leqslant \sum_{k \geqslant |t - \tilde{t}| + 1} \bar{s}^k \exp(-\tfrac{3}{2} p J_\pi k) \leqslant M_1 \exp[-\beta J_\pi(|t - \tilde{t}| + 1)] \end{aligned} \tag{71}$$

The estimate of the lemma follows from (70) and (71).

Proof of Theorem 2. Let $x \in Z^3$, $\varphi_\pi(x) = -1$, and $x \in A_t$. It follows from the definition of the regular ceilings that if A_t is a cube of the regular ceiling, then $\varphi(x) = -1$. Using clusters (Definition 3), Lemma 14, and the standard method for proving the existence of a phase transition in the Ising ferromagnet (Peierls estimate for pure boundary condition), one can show that

$$\text{Prob}(\varphi(x) = -1) > 1 - g(\beta) \tag{72}$$

where $g(\beta) \to 0$ when $\beta \to \infty$. The estimate (72) has been obtained for the volume $S_\Delta^- \setminus \bar{S}_\Delta$. The second estimate (5) can be proved analogously.

8. CONCLUDING REMARKS

1. The exponent g in Theorem 2 is chosen for the sake of concreteness. It follows easily from the proof of Theorem 2 that it can be generalized to other potentials. It is important only whether the conditions (i)–(vi) in Section 4 can be satisfied when β is large. Increasing the potential decay, we can improve the critical temperature value.

2. It would be extremely interesting to investigate the Gibbs state corresponding to the boundary conditions $\bar{\varphi}(x) = \{\varphi_\pi(x), x \in Z^3 \setminus W_{L,M}\}$ in the case of a nonrational plane π.

3. By analogy with ref. 3 or ref. 13 one can prove Theorem 2 for the model with the following finite-range interaction Hamiltonian in the case that the plane π is rational:

$$H_\pi(\varphi) = -\tfrac{1}{2} \sum_{\substack{x,\, y \in Z^3, \text{ and } (x,\, y) \\ \pi\text{-nearest}}} J(x-y)\, \varphi(x)\, \varphi(y) \tag{73}$$

The configuration $\varphi_\pi(x)$ is a stable ground state of model (73). Adding ferromagnetic interactions $J(x-y)$ (where x and y are not π-nearest) requires a larger Peierls constant for the ground state $\varphi_n(x)$. Despite this, all the techniques developed in Section 4 were focused exactly on proving that "extra" interactions do not invalidate Theorem 2. It would be interesting to prove the last assertion by the correlation inequalities method (see ref. 13).

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